ΘΕΩΡΗΜΑ (Αλγεβρα Ιακωβιανων Πινάκων)

Έστω σύνολο $U \subseteq \mathbb{R}^n$ ανοιχτό με $\bar{f}, \bar{g} : U \to \mathbb{R}^m$
και $\varphi, \psi : U \to \mathbb{R}$ μερικώς διαφορίσιμες
$\forall\ \psi(\bar{x}) \neq 0$ στο $\bar{x} \in U$. Τότε η $\bar{f} + \bar{g} : U \to \mathbb{R}^m$
και $\bar{f} \cdot \bar{g} : U \to \mathbb{R}$ και $\varphi \bar{f} : U \to \mathbb{R}$ και $\dfrac{\bar{f}}{\psi} : U \to \mathbb{R}^m$

$$J_{\bar{f}+\bar{g}}(\bar{x}) = J_{\bar{f}}(\bar{x}) + J_{\bar{g}}(\bar{x}) \in \mathbb{R}^{m \times n} \text{ και}$$

$$\mathrm{grad}(\bar{f} \cdot \bar{g})(\bar{x}) = \bar{f}(\bar{x})^T \cdot J_{\bar{g}}(\bar{x}) + \bar{g}(\bar{x})^T \cdot J_{\bar{f}}(\bar{x})$$

$$\Downarrow$$

$$(f_1(\bar{x}), \dots, f_m(\bar{x}))$$

όμοια, $J_{\varphi \bar{f}}(\bar{x}) = \varphi(\bar{x})\, J_{\bar{f}}(\bar{x}) + \bar{f}(\bar{x}) \cdot \mathrm{grad}\,\varphi(\bar{x}) \in \mathbb{R}^{n \times m}$

Απόδειξη

για την $\varphi\cdot\bar{f}$ με $\bar{f}=\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$

άρα $\varphi\cdot\bar{f}=\begin{pmatrix} \varphi\cdot f_1 \\ \vdots \\ \varphi\cdot f_m \end{pmatrix}$ και μετά εξ ορισμού

$$J\varphi\bar{f}(\bar{x})=\begin{pmatrix} \frac{\partial(\varphi f_1)}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial(\varphi f_1)}{\partial x_n}(\bar{x}) \\ \vdots & & \vdots \\ \frac{\partial(\varphi f_m)}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial(\varphi f_m)}{\partial x_n}(\bar{x}) \end{pmatrix}=$$

$$=\begin{pmatrix} \left(\frac{\partial\varphi}{\partial x_1}(\bar{x})\right)f_1(\bar{x})+\varphi(\bar{x})\frac{\partial f_1}{\partial x_1}(\bar{x}) & \cdots \\ \vdots & \vdots \end{pmatrix}=$$

$$=\begin{pmatrix} \varphi(\bar{x})\frac{\partial f_1}{\partial x_1}(\bar{x}) & \cdots & \varphi(\bar{x})\frac{\partial f_1}{\partial x_n}(\bar{x}) \\ \vdots & & \vdots \\ \varphi(\bar{x})\frac{\partial f_m}{\partial x_1}(\bar{x}) & \cdots & \varphi(\bar{x})\frac{\partial f_m}{\partial x_n}(\bar{x}) \end{pmatrix}+$$

$$+\begin{pmatrix} f_1(\bar{x})\cdot\frac{\partial\varphi}{\partial x_1}(\bar{x}) & \cdots & f_1(\bar{x})\frac{\partial\varphi}{\partial x_n}(\bar{x}) \\ \vdots & & \vdots \\ f_m(\bar{x})\cdot\frac{\partial\varphi}{\partial x_1}(\bar{x}) & \cdots & f_m(\bar{x})\frac{\partial\varphi}{\partial x_n}(\bar{x}) \end{pmatrix}=$$

$$=\varphi(\bar{x})\cdot J\bar{f}(\bar{x})+\bar{f}(\bar{x})\cdot\operatorname{grad}\varphi(\bar{x})\in\mathbb{R}^{n\times m}$$

ΕΧΟΥΜΕ:

- $f : \mathbb{R} \to \mathbb{R}$ με $f'(x) \in \mathbb{R}$
- $f : \mathbb{R}^n \to \mathbb{R}$ με $\operatorname{grad} f(\bar{x}) = \nabla f(\bar{x}) = \left(\frac{\partial f}{\partial x_1}(\bar{x}), \ldots, \frac{\partial f}{\partial x_n}(\bar{x}) \right) \in \mathbb{R}^n$

  Η ονομαζόμενη κλίση $(n \in \mathbb{N})$
- $f : \mathbb{R}^n \to \mathbb{R}^m$ με

$$J_{\bar{f}}(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{x}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\bar{x}) \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$(m \in \mathbb{N}, n \in \mathbb{N})$